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NONLINEAR FREE VIBRATION OF HEATED ORTHOTROPIC RECTANGULAR PLATES

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Abstract—An analytical analysis of free vibrations of a heated orthotropic rectangular thin plate under various boundary conditions is presented. The nonlinear governing equations are derived from von Kármán plate theory and Berger's analysis separately; from them the Duffing-type nonlinear ordinary equations are then obtained by employing Galerkin's method using one-term approximation. The methods of successive approximation and complete elliptic cosine are applied to solve the nonlinear equations. The influence of temperature changes and large amplitudes on the period of free vibrations are established; also the buckling temperature is obtained. The analytical solutions are compared with numerical results from Runge–Kutta method. Two different approaches to linearize thermoelastic plate equations are considered and compared.

NOTATION

a, b, h length, width and thickness of the rectangular plate, respectively.
e_1, e_2 the first and second strain invariant of midplane of isotropic plate, respectively.
E total energy of plate.
E_1, E_2 Young's moduli for an orthotropic material.
G_{12} shear modulus.
$M^T = \dfrac{12}{h^2}\displaystyle\int_{-h/2}^{h/2} \Delta T \, dz.$
$N^T = \dfrac{1}{h}\displaystyle\int_{-h/2}^{h/2} z\Delta T \, dz.$
t time.
T temperature.
T_0 a reference temperature.
$\Delta T = T - T_0$
T_N, T_L nonlinear and linear period of plate vibration, respectively.
u, v, W displacements.
x, y, z Cartesian coordinates.
z_0 initial thermal deflection $z_0 = z_0(T)$.
z_1, z_2 amplitudes on upper and lower directions of plate, respectively.
α coefficient of linear thermal expansion for isotropic material.
α_x, α_y coefficients of linear thermal expansion for an orthotropic material.
$\epsilon_{ij}, \sigma_{ij}$ strain and stress components, respectively.
λ_1 a frequency parameter.
ν_{12}, ν_{21} Poisson's ratios for an orthotropic material.
ρ plate density.
τ dimensionless time.
ω_L frequency of plate linear vibration at $\Delta T = 0$.
ω_N frequency of plate nonlinear vibration
$(\dot{\ }) = \dfrac{\partial}{\partial t} = (\ ),_t$ or $= \dfrac{\partial}{\partial \tau}$.
$(\ ),_x = \dfrac{\partial}{\partial x}$.

1. INTRODUCTION

Many thermoelastic problems have been treated by Nowacki[1] and Boley and Weiner[2]. Thermomechanically coupled vibrations of isotropic elastic plates were investigated by Nowacki[1], Čukić[3] and Chang[4]. Uncoupled vibrations were investigated by Sunakawa[5], Pal[6] and Bailey[7]. However, there are only a few investigations in the literature dealing with thermal stresses in orthotropic plates. Chang[8] has done a

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coupled analysis for an orthotropic rectangular plate. Bailey[9] has considered uncoupled linear vibration of heated orthotropic rectangular plate both numerically and experimentally. Pal[10] has analyzed uncoupled nonlinear vibrations of an orthotropic circular plate based on Berger's approximation method[11].

The purpose of this work is to present analytical methods to investigate the influence of temperature field on the free vibration of thermally stressed orthotropic rectangular plate with various boundary conditions. It is known that in linear uncoupled thermoelastic analysis (small deformation) the effect of temperature is treated as a body force[2], thus it cannot affect the natural frequencies of free vibration of an elastic body. And the only influence of temperature field in such an analysis is limited to forced vibration. Therefore, only either the nonlinear or the coupled linear analysis can be used to investigate the problem of thermal influence on plate free vibration. However, in uncoupled free vibration analysis, generally we have to take temperature field independent of time, so as to avoid a forced vibration problem.

In this paper, only the uncoupled case will be studied. And to avoid confusion in what follows, the terms "coupled" and "uncoupled" will be restricted to represent the coupling between the three displacement components u, v, W of the plate, unless otherwise specified for interdependence of temperature and displacement fields.

Here the following methods are employed to study the proposed problem:

(i) Von Kármán's nonlinear plate theory is used to derive three coupled governing equations for the plate vibration, by using Galerkin's technique and method of successive approximation to obtain a one-term approximate solution.
(ii) Berger's nonlinear plate analysis method is used to derive uncoupled quasi-linear equations for the plate, which is solved by using the same method as in case (i).
(iii) A method of direct linearization of von Kármán's theory is used to obtain a coupled linear solution.
(iv) A method of further uncoupling the linearized equations in (iii) is used to obtain a single linear equation for plate flexural vibration.

2. PROBLEM DESCRIPTION

Consider a rectangular thin plate with thickness h and edge lengths a and b. The midplane of the plate coincides with xy plane of an orthogonal Cartesian coordinate system. The following assumptions are made:

(i) The plate is made of orthotropic linear elastic material, obeys generalized Hooke's law; the plate geometrical axes of symmetry and the material elastic axes of symmetry coincide.
(ii) All the material properties are independent of temperature and are taken as constants.
(iii) The transverse shear, rotatory inertia and in-plane longitudinal inertia of the plate are neglected.
(iv) The plate is very thin, so the lateral displacement of the plate can be represented by the lateral displacement W of the midplane.
(v) Temperature field is assumed to be a known function and satisfies appropriate thermal boundary conditions.
(vi) The plate is intially bent due to thermal stresses and the initial deflection at center of the plate is z_0; z_1 and z_2 denote upward $(+z)$ and downward $(-z)$ central amplitude, respectively. When the initially heated deflection z_0 equals zero, the upward and downward amplitudes are equal $(z_1 = z_2)$.

The stress–strain constitutive relation for orthotropic material at plane-stress state is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_1/\mu & E_1\nu_{21}/\mu & 0 \\ & E_2/\mu & 0 \\ \text{symm.} & & G_{12} \end{bmatrix} \begin{bmatrix} \epsilon_x - \alpha_x\Delta T \\ \epsilon_y - \alpha_y\Delta T \\ 2\epsilon_{xy} \end{bmatrix}, \tag{1}$$

where $\mu = 1 - \nu_{21}\nu_{12}$, $E_1\nu_{21} = E_2\nu_{12}$.

For large-deformation plate analysis, the strain–displacement relations are

$$\begin{aligned}\epsilon_x &= \epsilon_x^0 - zW_{,xx} = u_{,x} + \tfrac{1}{2}W_{,x}^2 - zW_{,xx},\\ \epsilon_y &= \epsilon_y^0 - zW_{,yy} = v_{,y} + \tfrac{1}{2}W_{,y}^2 - zW_{,yy},\\ \epsilon_{xy} &= \epsilon_{xy}^0 - zW_{,xy} = (u_{,y} + v_{,x} + W_{,x}W_{,y}) - zW_{,xy},\end{aligned} \tag{2}$$

where ϵ_x^0, ϵ_y^0, ϵ_{xy}^0 are strain components on the midplane.

The boundary conditions to be satisfied for different cases are

(i) all-edge simply supported (s)

$$\begin{aligned}&W = 0, \quad u = 0, \quad \sigma_{xy} = 0,\\ &W_{,xx} + \nu_{21}W_{,yy} + (\alpha_x + \nu_{21}\alpha_y)M^T = 0 \quad \text{at} \quad x = 0, a,\\ &W = 0, \quad v = 0, \quad \sigma_{xy} = 0,\\ &W_{,xx} + \nu_{12}W_{,yy} + (\alpha_y + \nu_{12}\alpha_x)M^T = 0 \quad \text{at} \quad y = 0, b;\end{aligned} \tag{3}$$

(ii) all-edge clamped (c)

$$\begin{aligned}&W = 0, \quad u = 0, \quad \sigma_{xy} = 0, \quad W_{,x} = 0 \quad \text{at} \quad x = 0, a\\ &W = 0, \quad v = 0, \quad \sigma_{xy} = 0, \quad W_{,y} = 0 \quad \text{at} \quad y = 0, b;\end{aligned} \tag{4}$$

(iii) simply supported on a pair of opposite edges and remaining edges clamped (Mixed, M)

$$\begin{aligned}&W = 0, \quad u = 0, \quad \sigma_{xy} = 0\\ &W_{,xx} + \nu_{21}W_{,yy} + (\alpha_x + \nu_{21}\alpha_y)M^T = 0, \quad \text{at} \quad x = 0, a,\\ &W_{,y} = 0, \quad W = 0, \quad v = 0, \quad \sigma_{xy} = 0, \quad \text{at} \quad y = 0, b.\end{aligned} \tag{5}$$

3. NONLINEAR ANALYSIS

3.1. *Von Kármán plate theory*

From von Kármán's nonlinear plate theory, the governing equations for free vibration of a thermoelastic plate are

$$\bar{u}_{,\xi\xi} + C_1\lambda^2\bar{u}_{,\eta\eta} + C_2\lambda\bar{v}_{,\xi\eta} - L_1\bar{N}^T_{,\xi} = -\delta^2(\overline{W}_{,\xi\xi} + C_1\lambda^2\overline{W}_{,\eta\eta})\overline{W}_{,\xi} - C_2\delta^2\lambda^2\overline{W}_{,\xi\eta}\overline{W}_{,\eta}, \tag{6}$$

$$C_1\bar{v}_{,\xi\xi} + C_3\lambda^2\bar{v}_{,\eta\eta} + C_2\lambda\bar{u}_{,\xi\eta} - L_2\lambda\bar{N}^T_{,\eta} = -\delta^2(C_1\overline{W}_{,\xi\xi} + C_3\lambda^2\overline{W}_{,\eta\eta})\overline{W}_{,\eta}\lambda - C_2\lambda\delta^2\overline{W}_{,\xi\eta}\overline{W}_{,\xi}, \tag{7}$$

$$\begin{aligned}\overline{W}_{,\xi\xi\xi\xi} + 2C_4\lambda^2\overline{W}_{,\xi\xi\eta\eta} + C_3\lambda^4\overline{W}_{,\eta\eta\eta\eta} + \overline{W}_{,\tau\tau} = -(1/\delta^2)(L_1\overline{M}^T_{,\xi\xi} + L_2\lambda^2\overline{M}^T_{,\eta\eta})\\ - (12/\delta^2)(L_1\bar{N}^T W_{,\xi\xi} + L_2\lambda^2\bar{N}^T\overline{W}_{,\eta\eta}) + (12/\delta^2)(\bar{u}_{,\xi} + (\delta^2/2)\overline{W}^2_{,\xi})(\overline{W}_{,\xi\xi}\\ + \nu_{21}\lambda^2\overline{W}_{,\eta\eta}) + (12/\delta^2)\left(\lambda\bar{v}_{,\eta} + \frac{\delta^2\lambda^2}{2}\overline{W}^2_{,\eta}\right)(\nu_{21}\overline{W}_{,\xi\xi} + C_3\lambda^2\overline{W}_{,\eta\eta})\\ + (24C_1/\delta^2)(\lambda\bar{u}_{,\eta} + \bar{v}_{,\xi} + \delta^2\lambda\overline{W}_{,\xi}\overline{W}_{,\eta})\lambda\overline{W}_{,\xi\eta},\end{aligned} \tag{8}$$

where the following relations and dimensionless terms are used:

$$C_1 = \frac{\mu G_{12}}{E_1}, \quad C_2 = \nu_{21} + C_1, \quad C_3 = K^2 = \frac{E_2}{E_1},$$
$$C_4 = \nu_{21} + 2C_1, \quad \xi = x/a, \quad \eta = y/b, \quad \delta = h/a,$$
$$\lambda = a/b, \quad \overline{W} = W/h, \quad \overline{u} = u/a, \quad \overline{v} = v/a, \tag{9}$$
$$l_1 = \alpha_x + \nu_{21}\alpha_y, \quad l_2 = \nu_{21}\alpha_x + C_3\alpha_y, \quad L_1 = l_1 T_0,$$
$$L_2 = l_2 T_0, \quad \overline{N}^T = N^T/T_0, \quad \overline{M}^T = M^T/T_0,$$
$$C_p^2 = \frac{D_x}{\rho h a^2}, \quad \tau = C_p t, \quad D_x = \frac{E_1 h^3}{12\mu},$$

and the temperature functions $\overline{M}^T$ and $\overline{N}^T$ are given.

Assuming the temperature field to be symmetrical, we may expand the given spatial functions $\overline{N}^T$ and $\overline{M}^T$ in double Fourier series

$$\overline{N}^T(\xi, \eta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{b}, \tag{10}$$

$$\overline{M}^T(\xi, \eta) = \sum_{m,n=\text{odd}}^{\infty} \sum^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{11}$$

The flexural displacement $\overline{W}(\xi, \eta, \tau)$ could be described to be of a separable form by

$$\overline{W}(\xi, \eta, \tau) = \phi(\xi, \eta)\overline{W}_1(\tau), \tag{12}$$

where the shape function $\phi(\xi, \eta)$ must satisfy plate boundary conditions, and $\overline{W}_1(\tau)$ is to be determined. The shape function $\phi(\xi, \eta)$ for different boundary conditions may be assumed as

(i) all-edge simply supported (s)

$$\phi(\xi, \eta) = \sin \pi\xi \sin \pi\eta; \tag{13}$$

(ii) all-edge clamped (c)

$$\phi(\xi, \eta) = \sin^2 \pi\xi \sin^2 \pi\eta; \tag{14}$$

(iii) mixed (M)

$$\phi(\xi, \eta) = \sin \pi\xi \sin^2 \pi\eta. \tag{15}$$

One notes that all nonlinearities in eqns (6) and (7) are on the right-hand side, thus if $\overline{W}$ is treated as known function, then eqns (6) and (7) become linear equations for $\overline{u}$ and $\overline{v}$. And there exist two solutions for $\overline{u}$ and $\overline{v}$; a static solution due to input of $\overline{N}^T$, and a quasi-static one due to input $\overline{W}$. We will now proceed with all-edge simply supported plate to demonstrate the solution process.

To satisfy the boundary conditions, the displacement $\overline{u}$ has to be even (odd) function in $\eta(\xi)$ with respect to $\eta = \frac{1}{2}$ ($\xi = \frac{1}{2}$) while $\overline{v}$ has to be an even (odd) function in $\xi(\eta)$ with respect to $\xi = \frac{1}{2}(\eta = \frac{1}{2})$[12], and therefore the static solutions $\overline{u}_s$ and $\overline{v}_s$ can be described by

$$\overline{u}_s(\xi, \eta) = \sum \sum u_{mn} \sin 2m\pi\xi \cos 2n\pi\eta \qquad m = 1, 2, \ldots, \qquad n = 0, 1, 2, \ldots, \tag{16}$$

$$\overline{v}_s(\xi, \eta) = \sum \sum v_{mn} \cos 2\pi m\xi \sin 2\pi n\eta \qquad m = 0, 1, 2, \ldots, \qquad n = 1, 2, \ldots. \tag{17}$$

Inserting eqn (10) and eqns (16) and (17) into eqns (6) and (7), u_{mn} and v_{mn} are solved as

$$u_{mn} = \frac{(A_{mn}/2\pi)[L_1 m(C_1 m^2 + C_3\lambda^2 n^2) - C_2\lambda^2 L_2 mn^2]}{(m^2 + C_1\lambda^2 n^2)(C_1 m^2 + C_3\lambda^2 n^2) - C_2^2\lambda^2 m^2 n^2}, \tag{18}$$

$$v_{mn} = \frac{(A_{mn}/2\pi)[L_1\lambda n(m^2 + C_1\lambda^2 n^2) - C_2\lambda L_1 m^2 n]}{(m^2 + C_1\lambda^2 n^2)(C_1 m^2 + C_3\lambda^2 n^2) - C_2^2\lambda^2 m^2 n^2}. \tag{19}$$

The quasi-static part of eqns (6) and (7) are of the form

$$\begin{aligned}\bar{u}_{,\xi\xi} + C_1\lambda^2\bar{u}_{,\eta\eta} + C_2\lambda\bar{v}_{,\xi\eta} = {} & -(\pi^3/4)\delta^2\overline{W}_1^2(\tau)(1 + C_1\lambda^2)\sin 2\pi\xi\cos 2\pi\eta \\ & - (\pi^3/4)C^2\delta^2\lambda^2\overline{W}_1^2(\tau)\sin 2\pi\xi\cos 2\pi\eta \\ & + (\pi^3/4)\delta^2\overline{W}_1^2(\tau)(1 + C_1\lambda^2 - C_2\lambda^2)\sin 2\pi\xi,\end{aligned} \tag{20}$$

$$\begin{aligned}C_1\bar{v}_{,\xi\xi} + C_3\lambda^2\bar{v}_{,\eta\eta} + C_2\lambda\bar{u}_{,\xi\eta} = {} & (\pi^3/4)\delta^2\overline{W}_1^2(\tau)(C_3\lambda^3 + C_1\lambda - C_2\lambda)\sin 2\pi\eta \\ & - (\pi^3/4)\delta^2\overline{W}_1^2(\tau)(C_3\lambda^3 + C_1\lambda + C_2\lambda)\sin 2\pi\eta\cos 2\pi\xi.\end{aligned} \tag{21}$$

Using method of undetermined coefficient, we may obtain the quasi-static solutions

$$\bar{u}_d(\xi, \eta, \tau) = \frac{\pi\overline{W}_1^2(\tau)}{16}\delta^2[\cos 2\pi\eta + \nu_{21}\lambda^2 - 1]\sin 2\pi\xi, \tag{22}$$

$$\bar{v}_d(\xi, \eta, \tau) = \frac{\pi\overline{W}_1^2(\tau)}{16}\delta^2\left[\lambda\cos 2\pi\xi + \frac{\nu^{21}}{C_3\lambda} - \lambda\right]\sin 2\pi\eta. \tag{23}$$

Henceforth, the solutions for $\bar{u}$ and $\bar{v}$ are

$$\bar{u}(\xi, \eta, \tau) = \bar{u}_3 + \bar{u}_d, \qquad \bar{v}(\xi, \eta, \tau) = \bar{v}_3 + \bar{v}_d. \tag{24}$$

Next, we will insert the solutions $\bar{u}$ and $\bar{v}$ from eqn (24) and W into flexural vibration governing eqn (8), by using Galerkin's technique:

$$\int_R\int L_3(\bar{u}, \bar{v}, \overline{W})\phi(\xi, \eta)\,d_\xi\,d_\eta = 0 \tag{25}$$

to obtain the Duffing-type nonlinear ordinary differential equation

$$\begin{aligned}\ddot{W}_1(\tau) + \pi^4[1 + 2C_4\lambda^2 + C_3\lambda^4]\overline{W}_1(\tau) = {} & \tfrac{3}{8}\pi^4\overline{W}_1^3(\tau)\left[-3 + \frac{\nu_{21}^2}{C_3} - 4\nu_{21}\lambda^2\right. \\ & \left. + \lambda^4(\nu_{21}^2 - 3C_3)\right] + 6\pi^3\frac{\overline{W}_1(\tau)}{\delta^2}[1 + \nu_{21}\lambda^2][2u_{10} - u_{11}] - 6\pi^3\frac{\overline{W}_1(\tau)}{\delta^2} \\ & \times\lambda[\nu_{21} + C_3\lambda^2][2v_{01} - v_{11}] - 12\pi^3\frac{\overline{W}_1(\tau)}{\delta^2}[C_1\lambda^2 u_{11} + C_1\lambda v_{11}] \\ & + 3\pi^2\frac{\overline{W}_1(\tau)}{\delta^2}[L_1 + L_2\lambda^2][4A_{00} - 2A_{01} - 2A_{10} + A_{11}] + F_s,\end{aligned} \tag{26}$$

where

$$F_s = \frac{\pi^2}{\delta^2}\frac{L_1 B_{11}}{4} + \frac{\pi^2}{\delta^2}\frac{L_2^2\lambda^2}{4}B_{11}. \tag{27}$$

Similarly the single-mode equation for other boundary conditions will be Duffing-type equation of the same kind.

3.2. *Berger's method*

The strain energy for a thermoelastic orthotropic rectangular plate is of the form

$$U = \frac{1}{2}\int_A \left\{ (E_1 h/\mu)[e_1^{*2} - 2(k - \nu_{21})e_2^*] + \frac{E_1 h^3}{12\mu}\left[W_{,xx}^2 + C_3 W_{,yy}^2 + 2\nu_{21} W_{,xx} W_{,yy} + 4\frac{\mu G_{12}}{E_1} W_{,xy}^2 \right] - \frac{2E_1 h}{\mu}[l_1 \epsilon_x^0 + l_2 \epsilon_y^0] N^T + \frac{E_1 h^2}{6\mu}[l_1 W_{,xx} + l_2 W_{,yy}] M^T \right\} \mathrm{d}A, \tag{28}$$

where

$$e_1^* = \epsilon_x^0 + K\epsilon_y^0, \qquad e_2^* = \epsilon_x^0 \epsilon_y^0 - \frac{2G_{12}\mu}{E_1(K - \nu_{21})}\epsilon_{xy}^{02},$$

According to Berger's approximation approach for isotropic plate analysis, when the second strain invariant of the midplane $e_2 = \epsilon_x^0 \epsilon_y^0$ is neglected in the plate strain energy, decoupled simple quasi-linear equations can be obtained, which simplifies the analysis greatly. However, in orthotropic plate analysis, neglecting the true second strain invariant of the midplane e_2 from the strain energy does not provide similar simplification. In order to gain similar advantages of Berger's method, a quantity e_2^* is chosen for orthotropic plate, which will play the same role as e_2 in isotropic case[13]. Thus the Berger's governing equations for plate free vibration can be deduced from applying variation theorems as

$$D_x W_{,xxxx} + 2HW_{,xxyy} + D_y W_{,yyyy} + (12D_x/h^2)[-(e_1^* W_{,x})_{,x} + l_1(N^T W_{,x})_{,x}] + (12D_x/h^2)[-K(e_1^* W_{,y})_{,y} + l_2(N^T W_{,y})_{,y}] + (D_x/h)[l_1 M_{,xx}^T + l_2 M_{,yy}^T] + \rho h W_{,tt} = q(x, y, t), \tag{29}$$

$$-e_{1,x}^* + l_1 N_{,x}^T = 0, \tag{30}$$

$$-Ke_{1,y}^* + l_2 N_{,y}^T = 0, \tag{31}$$

where

$$D_x = E_1 h^3/12\mu, \qquad D_y = E_2 h^3/12\mu,$$
$$H = E_1 \nu_{21} h^3/12\mu + G_{12} h^3/6.$$

The major contribution of Berger's method for isotropic plate analysis is to yield an uncoupled quasi-linear governing equation for flexural amplitude W. When it is applied to thermoelastic orthotropic plate analysis, certain restrictions on eqns (29)–(31) must be made to acquire the similar benefit from Berger's method; such as

(i) $l_1 = l_2/K$, the isotropic case, or

(ii) N^T = constant.

Therefore, we will limit the use of Berger's method to taking N^T as a constant in our analysis.

After integrating eqns (29)–(31) over plate area, with use of boundary conditions and Green's theorem to get rid of the terms involving u and v; then through non-dimensionalization, we obtain a single nonlinear equation:

$$\overline{W}_{,\xi\xi\xi\xi} + 2(4\lambda^2 \overline{W}_{,\xi\xi\eta\eta} + C_3\lambda^4 \overline{W}_{,\eta\eta\eta\eta} + \overline{W}_{,\tau\tau} + (12/\delta^2)[-e_1^* + L_1\overline{N}^T]\,\overline{W}_{,\xi\xi} + \frac{12\lambda^2}{\delta^2}[-Ke_1^* + L_2\overline{N}^T]\overline{W}_{,\eta\eta} = -(1/\delta^2)[L_1\overline{M}_{,\xi\xi}^T + L_2\lambda^2\overline{M}_{,\eta\eta}^T], \tag{32}$$

where

$$e_1^* = -(\delta^2/2)\int_0^1\int_0^1 \overline{W}[\overline{W}_{,\xi\xi} + K\lambda^2\overline{W}_{,\eta\eta}]\,d\xi\,d\eta$$

is a function of dimensionless time τ only.

In Berger's analysis, the same boundary conditions (3)–(5) and the same assumption (12)–(15) are used. Therefore, after application of Galerkin's method, the Duffing-type nonlinear differential equation is obtained for all-edge simply supported plate as

$$\ddot{W}_1(\tau) + \pi^4\left[1 + 2C_4\lambda^2 + C_3\lambda^4 - \frac{12\overline{N}^T}{\delta^2\pi^2}(L_1 + L_2\lambda^2)\right]\overline{W}_1(\tau) + (1 + K\lambda^2)\pi^4(3/2 + (3/2)K\lambda^2)\overline{W}_1^3(\tau) = 4F_s. \quad (33)$$

For other boundary conditions, the similar Duffing-type equations can be produced.

3.3. *Method of solution*

It is not found that, regardless of boundary conditions, single-mode analysis reduces both von Kármán's and Berger's equations to a Duffing-type equation of the general form

$$\ddot{W}_1(\tau) + \alpha_1\overline{W}_1(\tau) + \alpha_3\overline{W}_1^3(\tau) = F, \quad (34)$$

where α_1, α_3 and F are known constants.

Consider $\overline{W}_1(\tau)$ as sum of the displacement $z_0(T)$ due to the temperature change and the isothermal amplitude of vibration $\bar{z}(\tau)\,|_{\text{isothermal}}$, as in [5, 6].

$$\overline{W}_1(\tau) = z_0(T) + \bar{z}(\tau)\,|_{\text{isothermal}}. \quad (35)$$

Inserting $\overline{W}(\tau)$ into eqn (34), two equations are then obtained: the equation corresponding to the vibration state is

$$\ddot{\bar{z}}(\tau) + (\alpha_1 + 3\alpha_3 z_0^2)\bar{z}(\tau) + 3\alpha_3 z_0\bar{z}^2(\tau) + \alpha_3\bar{z}^3(\tau) = 0, \quad (36)$$

and the equation corresponding to the deflection state due to the temperature change is

$$\alpha_1 z_0 + \alpha_3 z_0^3 = F. \quad (37)$$

Equation (37) can be solved directly for static solution z_0.

Using the relation

$$\Lambda = \sqrt{\alpha_1 + 3\alpha_3 z_0^2}\,\tau, \quad (38)$$

eqn (36) is reduced to

$$d^2\bar{z}/d\Lambda^2 + \bar{z} + f_2\bar{z}^2 + f_3\bar{z}^3 = 0, \quad (39)$$

where

$$f_2 = \frac{3\alpha_3 z_0}{\alpha_1 + 3\alpha_3 z_0^2}, \qquad f_3 = \frac{\alpha_3}{\alpha_1 + 3\alpha_3 z_0^2}.$$

By using the transformation

$$\theta = \sqrt{1 + \beta}\,\Lambda, \quad (40)$$

eqn (39) is transformed into

$$(1 + \beta)\frac{d^2\bar{z}}{d\theta^2} + \bar{z} = -f_2\bar{z}^2 - f_3\bar{z}^3. \tag{41}$$

Let z_1 and $-z_2$ be the maximum and minimum amplitude, respectively, of the displacement $\bar{z}$; then, β and $\bar{z}$ are expanded in the power series of z_2 as

$$\beta = -\beta_1 z_2 + \beta_2 z_2^2 - \beta_3 z_2^3 + \cdots, \tag{42}$$

$$\bar{z} = -\Omega_1(\theta)z_2 + \Omega_2(\theta)z_2^2 - \Omega_3(\theta)z_2^3 + \Omega_4(\theta)z_2^4 \ldots. \tag{43}$$

Substituting eqns (42) and (43) in eqn (41), and using successive approximation method, under the initial conditions

$$\Omega_1(0) = 1, \qquad \Omega_2(0) = \Omega_3(0) = \Omega_4(0) = \cdots = 0,$$
$$\dot{\Omega}_1(0) = \dot{\Omega}_2(0) = \dot{\Omega}_3(0) = \dot{\Omega}_4(0) = \cdots = 0,$$

final form becomes

$$\begin{aligned}
\bar{z}(\theta) = {} & \left[-\frac{1}{2}f_2z_2^2 + \frac{1}{3}f_2^2z_2^3 - \left(\frac{25}{48}f_2^4 - \frac{21}{32}f_2f_3\right)z_2^4 \right. \\
& \left. + \left(\frac{25}{36}f_2^4 - \frac{29}{24}f_2^2f_3\right)z_2^5 \ldots\right] + \left[-z_2 + \frac{1}{3}f_2z_2^2 - \left(\frac{29}{144}f_2^2 - \frac{1}{32}f_3\right)z_2^3\right. \\
& \left. + \left(\frac{119}{432}f_2^3 - \frac{35}{96}f_2f_3\right)z_2^4 - \left(\frac{7103}{20736}f_2^4 - \frac{1607}{2304}f_2^2f_3 + \frac{23}{1024}f_3^2\right)z_2^5 + \cdots\right]\cos\theta \\
& + \left[\frac{1}{6}f_2z_2^2 - \frac{1}{9}f_2^2z_2^3 + \left(\frac{2}{9}f_2^3 - \frac{1}{3}f_2f_3\right)z_2^4\right. \\
& \left. - \left(\frac{8}{27}f_2^4 - \frac{5}{9}f_2^2f_3\right)z_2^5 + \cdots\right]\cos 2\theta + \left[-\left(\frac{1}{48}f_2^2 + \frac{1}{32}f_3\right)z_2^3\right. \\
& \left. + \left(\frac{1}{48}f_2^3 + \frac{1}{32}f_2f_3\right)z_2^4 - \left(\frac{31}{576}f_2^4 + \frac{11}{384}f_2^2f_3 - \frac{3}{128}f_3^2\right)z_2^5 + \cdots\right]\cos 3\theta \\
& + \left[\left(\frac{1}{432}f_2^3 + \frac{1}{96}f_2f_3\right)z_2^4 - \left(\frac{1}{648}f_2^4 + \frac{1}{72}f_2^2f_3\right)z_2^5 + \cdots\right)\cos 4\theta \\
& + \left[-\left(\frac{5}{20736}f_2^4 + \frac{5}{2304}f_2^2f_3 + \frac{1}{1024}f_3^2\right)z_2^5 + \cdots\right]\cos 5\theta.
\end{aligned} \tag{44}$$

And the period of the motion is given by

$$\begin{aligned}
T^*(\theta) = 2\pi\left[1 + \left(\frac{5}{12}f_2^2 - \frac{3}{8}f_3\right)z_2^2 - \left(\frac{5}{18}f_2^3 - \frac{1}{4}f_2f_3\right)z_2^3\right. \\
\left. + \left(\frac{385}{576}f_2^4 - \frac{275}{192}f_2^2f_3 + \frac{57}{256}f_3^2\right)z_2^4 \ldots\right.
\end{aligned} \tag{45}$$

Equations (44) and (45) form the solutions for eqn (41), and are influenced by the initial deflection z_0 and initial amplitude z_2. Next, the relation between z_1 and z_2 can be established through energy conservation principle as

$$(d\bar{z}/d\Lambda)^2 + \bar{z}^2 + \tfrac{2}{3}f_2\bar{z}^3 + \tfrac{1}{2}f_3\bar{z}^4 = 2\bar{E}, \tag{46}$$

where $\bar{E}$ is the total energy of the vibrating system.

There exist extremum for $\bar{z}$ at $\bar{z} = z_1$ and $-z_2$, when $d\bar{z}/d\Lambda = 0$, and this condition reduces eqn (46) to

$$z_1^2(1 + \tfrac{2}{3}f_2z_1 + \tfrac{1}{2}f_3z_1^2) = z_2^2(1 - \tfrac{2}{3}f_2z_2 + \tfrac{1}{2}f_3z_2^2) = 2\bar{E}. \tag{47}$$

Equation (47) thus can be used to determine z_1 and z_2 independently whenever $\bar{E}$ is given in accordance with the initial conditions.

When $\bar{M}^T = 0$, eqn (37) gives $z_0 = 0$, and then eqn (36) can be reduced to a standard Duffing equation as

$$\ddot{\bar{z}} + \alpha_1\bar{z} + \alpha_3\bar{z}^3 = 0. \tag{48}$$

When initial conditions $\bar{z}(0) = \bar{A}$ and $\dot{\bar{z}}(0) = 0$, the solution of eqn (48) is obtained as

$$\bar{z}(\tau) = \bar{A}C_n(p\tau, k), \tag{49}$$

where C_n is complete elliptical cosine with

$$p = \sqrt{\alpha_1 + \bar{A}^2\alpha_3}, \qquad k = \sqrt{\frac{\bar{A}^2\alpha_3}{2(\alpha_1 + \bar{A}^2\alpha_3)}}.$$

Also, the ratio between nonlinear period and linear period of plate free vibration is established as

$$\frac{T_N}{T_L} = \frac{2\sqrt{\alpha_1}\,\bar{K}(k)}{\pi\sqrt{\alpha_1 + \bar{A}^2\alpha_3}}, \tag{50}$$

where $\bar{K}(k)$ is the complete elliptic integral of the first kind with parameter k.

Here, the critical buckling temperature for thermally stressed orthotropic plate will be introduced. Equation (34) is a so-called weak nonlinear differential equation, which means that the coefficient α_3 of nonlinear term influences plate vibration frequency less than the coefficient α_1 of linear term. When linear vibration is considered ($\alpha_3 = 0$), for positive values of $\alpha_1(\alpha_1 > 0)$, we will have harmonic vibration; whereas for negative values of $\alpha_1(\alpha_1 < 0)$, we will have hyperbolic solutions instead, i.e. the amplitude increases with time. Since α_1 is dependent on temperature, α_1 may be zero when certain temperature is reached, and we shall call this temperature as a critical temperature for the plate.

4. LINEAR ANALYSIS

4.1. *Linearized coupled solution*

When the nonlinear term in eqn (34) is deleted, a linearized coupled solution can be obtained by solving the linear free-vibration equation:

$$\ddot{\bar{W}}_1(\tau) + \alpha_1\bar{W}_1(\tau) = 0, \tag{51}$$

which indicates that the obtained solution $\bar{W}_1(\tau)$ is influenced by the in-plane displacements $\bar{u}$ and $\bar{v}$.

4.2. *Single linear equation (uncoupled)*

To delete the higher-order nonlinear terms from von Kármán's plate governing eqns (6)–(8) and further uncouple the interdependence of $\bar{u}$, $\bar{v}$ and $\bar{W}$, an uncoupled single linear differential equation for flexural displacement $\bar{W}$ can be established as

$$\bar{W}_{,\xi\xi\xi\xi} + 2C\lambda^2\bar{W}_{,\xi\xi\eta\eta} + C_3\lambda^4\bar{W}_{,\eta\eta\eta\eta} + \bar{W}_{,\tau\tau} = (-12/\delta^2)(L_1N^T\bar{W}_{,\xi\xi} + L_2\lambda^2N^T\bar{W}_{,\eta\eta}), \tag{52}$$

which is identical to those derived by Parkus[14] and used by Bailey[9]. In deriving eqn (52) we have dropped $\overline{M}^T$ term, to avoid thermally induced vibration. Equation (52) may be solved approximately by applying Galerkin's technique to obtain a single-mode solution.

Since eqn (52) is self-adjoint, with respect to appropriate boundary conditions, the fundamental frequency can be obtained from Rayleigh–Quotient; also all the frequencies can be determined by Rayleigh–Ritz method.

5 RESULTS AND DISCUSSIONS

For numerical results analysis, we use the material constants as listed in Table 1[15] and assume

$$\overline{N}^T(\xi, \eta) = T_m \sin(\pi\xi) \sin(\pi\eta). \tag{53}$$

The comparison of results from different linearized analyses [eqns (51) and (52)] are illustrated in Figs 1 and 2 for all-edge simply supported and all-edge clamped plate, respectively, where frequency parameter λ_1 is an indicator of linear vibrating frequency.

Table 1. Material properties [15]

C_1	C_2	C_3	C_4	ν_{12}	α_x	α_y
.164	.228	.256	.392	.064	.46α	3.07α

α - coefficient of linear thermal expansion for a reference isotropic material

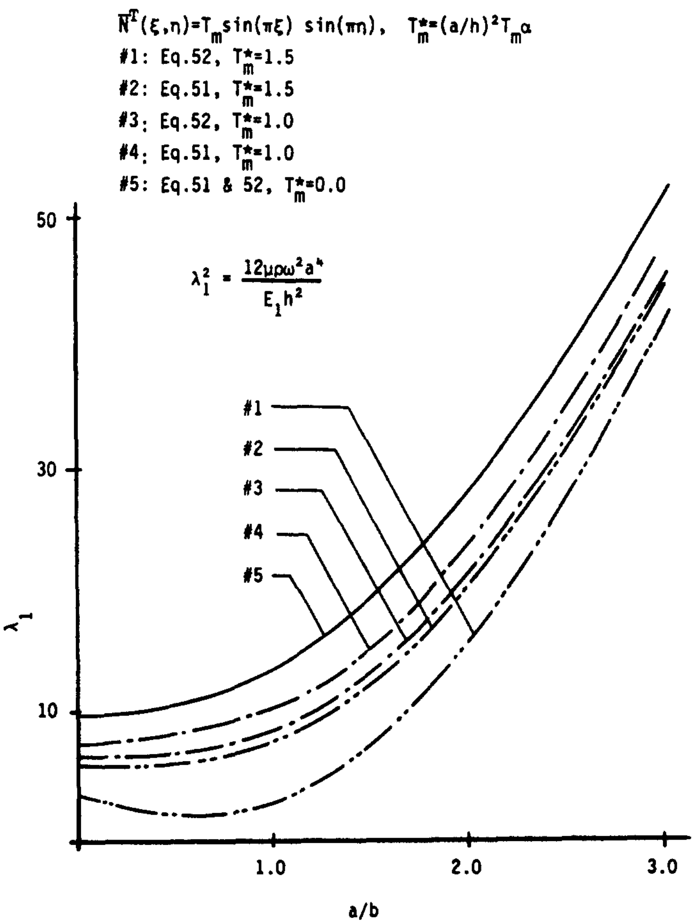

Fig. 1. All-edge simply supported plate

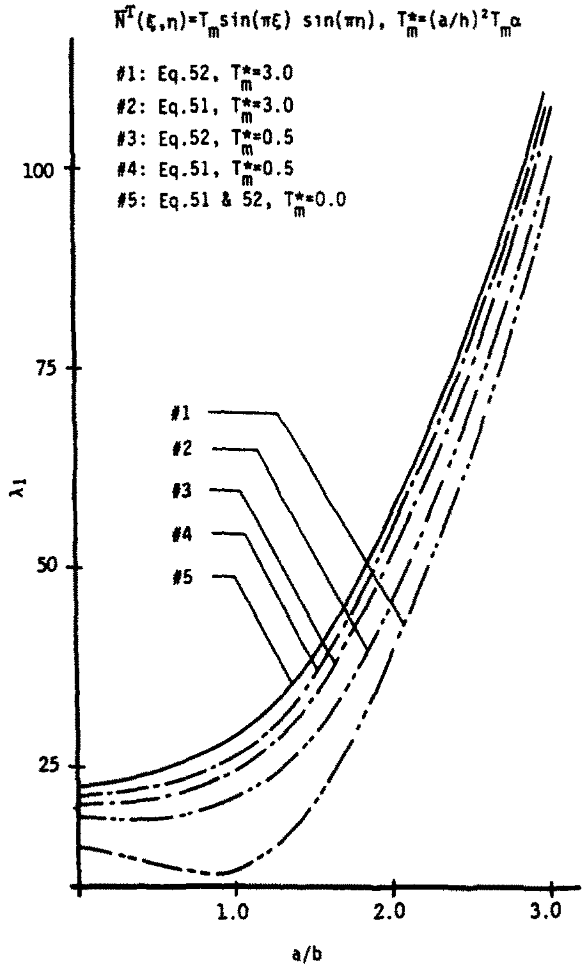

Fig. 2. All-edge clamped plate.

It is shown that at low temperatures, the two analyses offer close results; however, at higher temperature ($T_m^* = 1.5$) and when the aspect ratio is small ($a/b < 1.5$), there exist obvious differences between the two analyses. This behavior indicates that at higher temperature, the frequency of plate lateral motion is influenced by the in-plane displacements u and v significantly, and the coupling effect should not be neglected. Here we have done some calculations to find that this is not related to the shape function chosen or the approximation error.

Figure 3 shows variation of critical buckling temperature for both orthotropic and isotropic plates with different aspect ratio. It is observed that for isotropic plate, the temperature decreases when the aspect ratio increases, and the all-edge simply supported plate has lowest critical temperature. However, for orthotropic plate the critical temperature, in general, does not necessarily decrease with increasing aspect ratio; which is due to the anisotropic characteristics of material physical properties.

Under different edge conditions, the relation between nonlinear and linear period ratio T_N/T_L and amplitude $\bar{z}$ is illustrated in Fig. 4 for various temperatures. It is seen that the ratio T_N/T_L decreases with increasing amplitude $\bar{z}$, and temperature has the most significant effect on vibration period for all-edge simply supported plate.

Figure 5 shows for all-edge simply supported plate, under different initial thermal displacements z_0, the relation of nonlinear and linear frequency ratio and amplitude for various temperature distributions; where ω_L is measured at $T_m^* = 0$. For small initial displacement (z_0) and low temperature (T_m^*), the nonlinear frequency increases with amplitude; however for large initial displacement (z_0) and high temperature (T_m^*) the nonlinear frequency decreases as amplitude increases. When amplitude is small (linear case), frequency decreases with increasing temperature (#1, 2, 3)(#4, 5, 6, 7). And

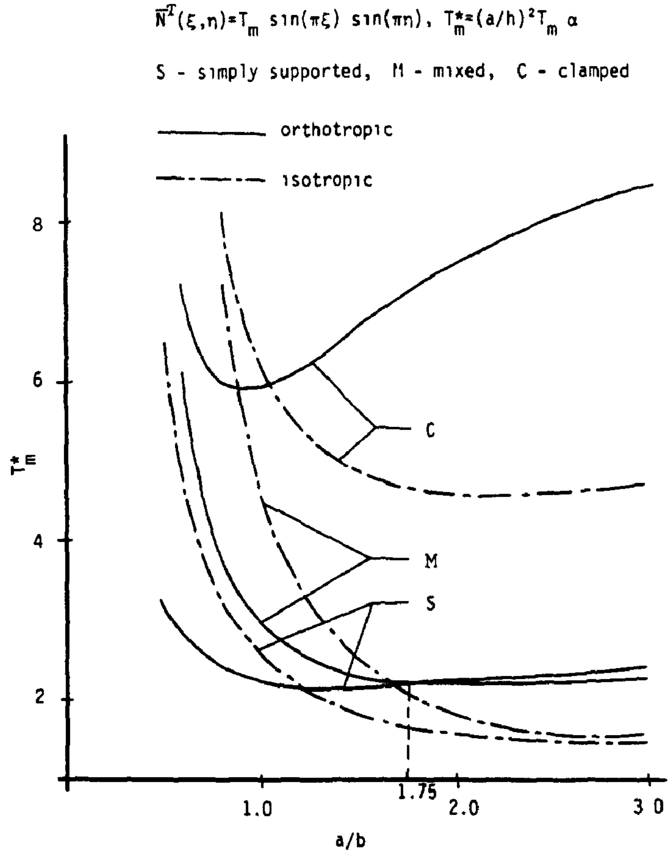

Fig. 3. Critical buckling temperature.

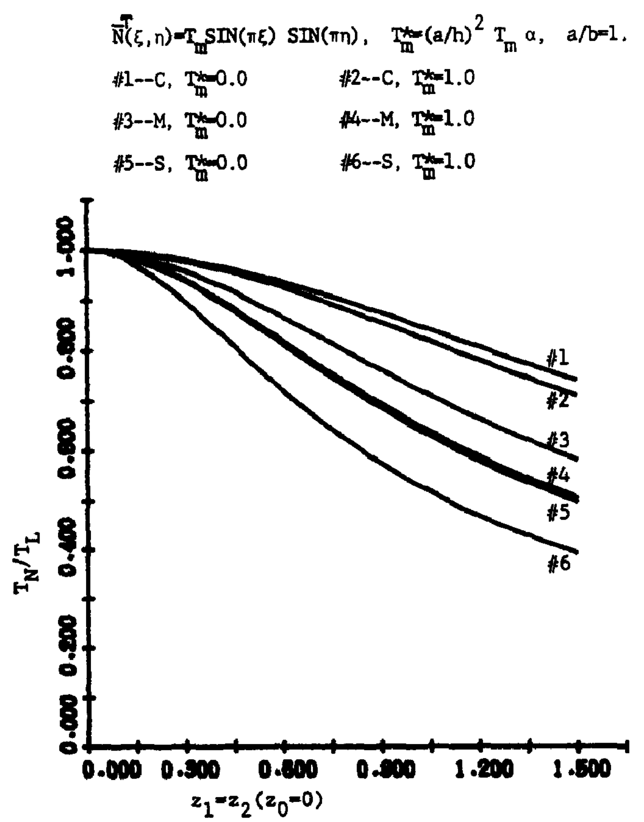

Fig. 4 Influence of large amplitude on T_N/T_L of plate.

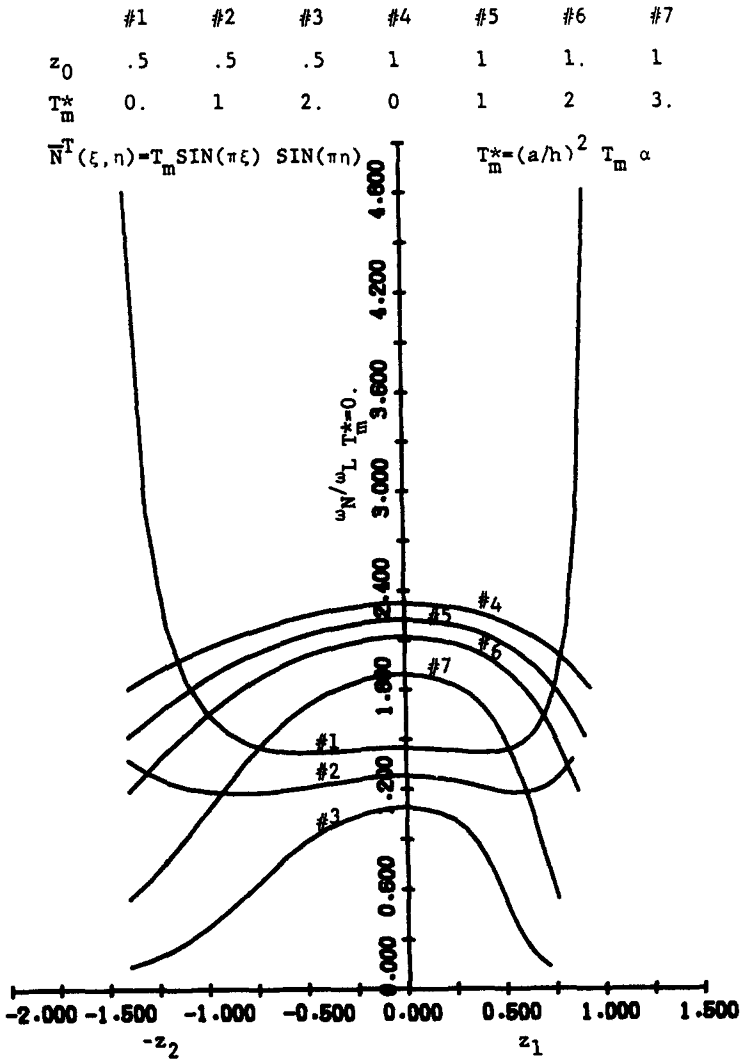

Fig. 5. Variation of frequency vibration of plate with temperature rise and for large-amplitude simply supported plate ($a/b = 1$)

Table 2. Results of T_N/T_L from different methods

$\ddot{\bar{Z}}(t) + a\bar{Z}(t) + b\bar{Z}^3(t) = 0,\quad \bar{Z}(0) = \dot{\bar{Z}}(0) = 0$

b/a	.3099	.4549	.5715	1.498	2.70	16.62
K(k)(Eq.50)	.9014	.8684	.8382	.6902	.5801	.2773
RUKU*	.8997	.8655	.8383	.6956	.5794	.2769

$\ddot{\bar{Z}}(t) + 1260\bar{Z}(t) + 927.9\bar{Z}^2(t) + 309.3\bar{Z}^3(t) = 0,\quad \bar{Z}(0) = \dot{\bar{Z}}(0) = 0$

z_1	.1	.2	.3	.4	.5	.6	.7	.8
z_2(Eq.47)	.10516	.22177	.35177	.49749	.66160	.84670	1.054	1.283
T*(θ)(Eq.45)	.17725	.17805	.17948	.18165	.18466	.18869	.1939	.2007
RUKU*	.1772	.1780	.1795	.1816	.1822	.1879	.1917	.1949

* Runge-Kutta Method

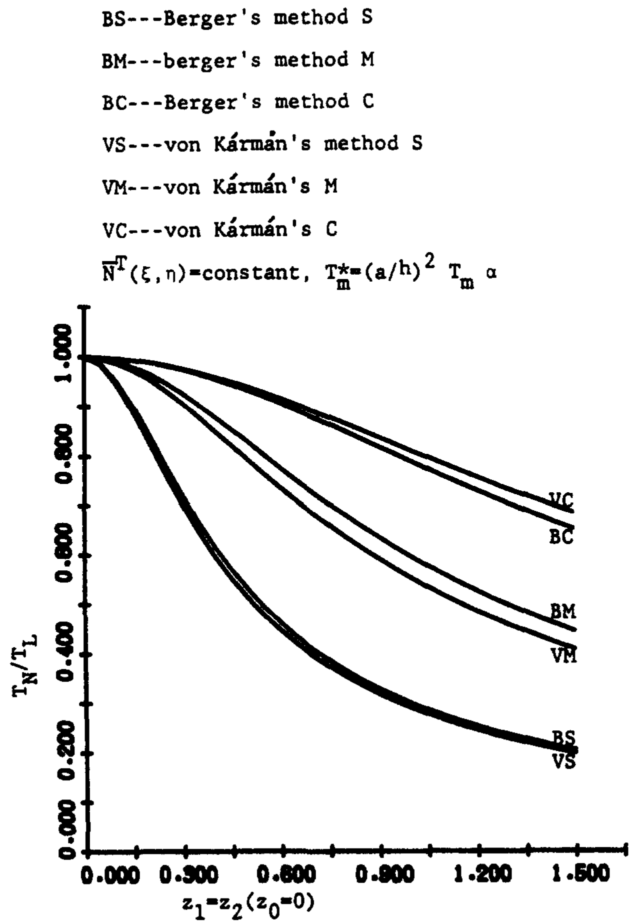

Fig. 6. Berger's method vs von Kármán's theory

for same temperature, distribution frequency increases with initial displacement (z_0) (#2 and #5) or (#3 and #6).

At constant temperature, we may compare the results of von Kármán's theory with that of Berger's method, as shown in Fig. 6, which shows that Berger's analysis gives satisfactory results.

Finally, in Table 2 we compare the analytical results for the ratio T_N/T_L from method of successive approximation [eqn (45)] and from complete elliptical cosine [eqn (50)] together with that obtained from numerical analysis method of Runge–Kutta[16], and we find they all are very close.

6. CONCLUSION

A method of one-term approximation solution is presented to study both the linear and nonlinear free-vibration behaviors of heated orthotropic rectangular plate under various boundary conditions. For nonlinear vibration analysis, fundamental equations of motion are derived from both von Kármán's and Berger's methods, and are solved by employing techniques of Galerkin and successive approximation. The linear solutions are obtained either from linearizing von Karman's solutions or from solving decoupled single linear equation.

Through our investigations in this paper, the following conclusions are drawn:

(i) Although von Kármán's analysis generally is quite complex to deal with, we have presented a relatively simple method to solve for nonlinear free vibration of heated orthotropic rectangular plate.

(ii) Berger's approach offers results in good agreement with that from von Kármán's theory, yet highly simplifies the analysis. However, for heated orthotropic rectangular-plate analysis, it is restricted to taking N^T = constant, otherwise the advantages of simplification will not be obtained.

(iii) In linearized analysis, the frequency of lateral vibration decreases with increasing temperature. At low temperatures, to use single linear equation (52) is not only simple but also gives good results. However, at high temperatures, it is suggested to use linearized von Kármán's solution so as to obtain satisfactory results.

(iv) Because of the anisotropic property of material constants, the critical temperature does not necessarily decrease with increasing plate aspect ratio. Of the three different edge conditions discussed, the all-edge simply supported plate has the lowest critical temperature.

(v) The ratio of nonlinear and linear period of plate vibration T_L/T_N decreases with increasing amplitude. Of all the three edge conditions discussed, the vibration period for all-edge simply supported plate is the most significantly affected by temperature.

(vi) With zero initial thermal deflection ($z_0 = 0$), the frequency of plate vibration increases with amplitude, and the nonlinear and linear period ratio T_N/T_L decreases with increasing temperature. After z_0 and T_m^* are increased to certain values, the frequency of plate vibration decreases with increasing amplitude. Also if the initial thermal deflection is not zero ($z_0 \neq 0$), then $z_1 \neq z_2$.

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